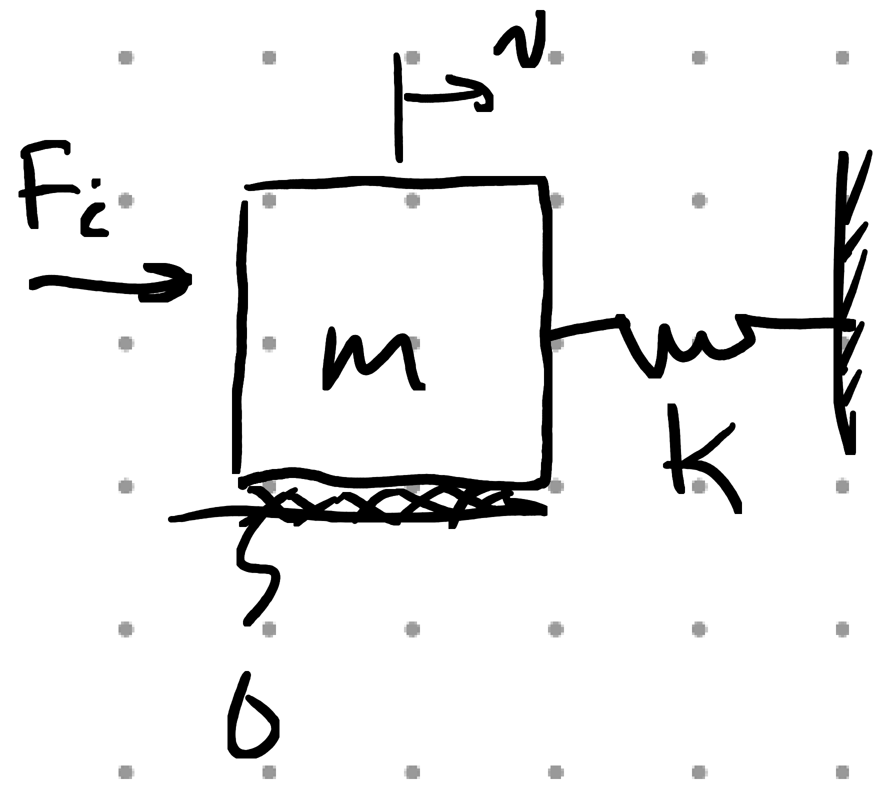
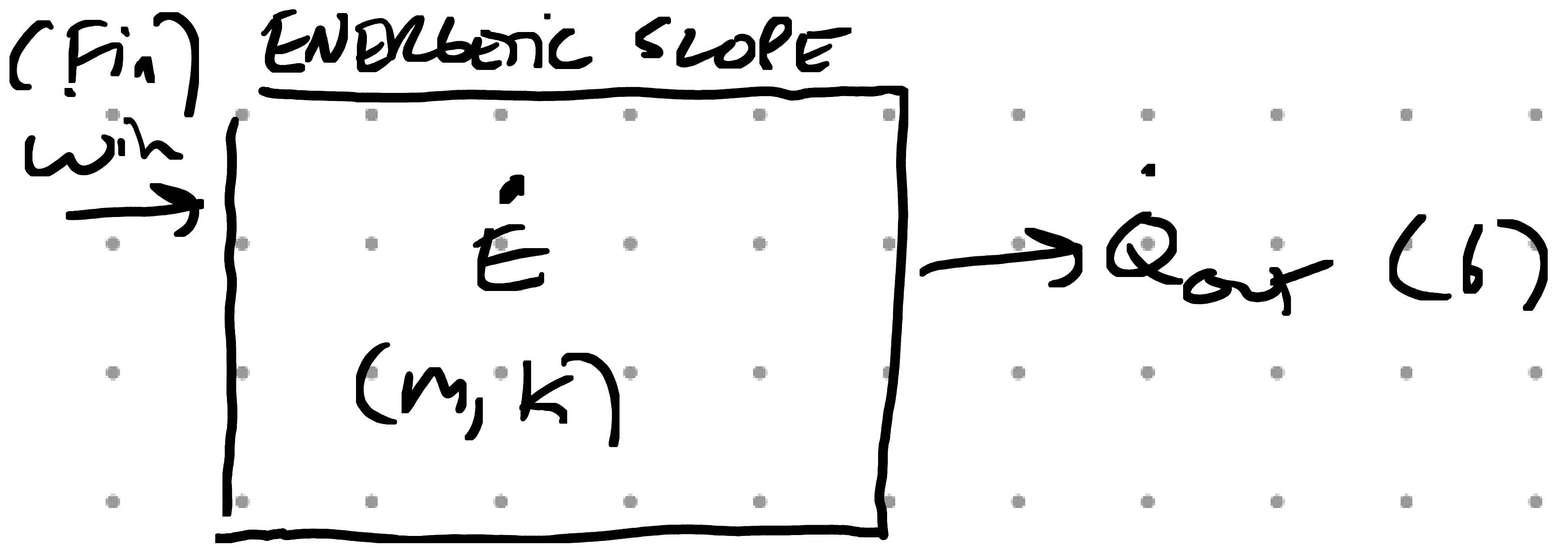
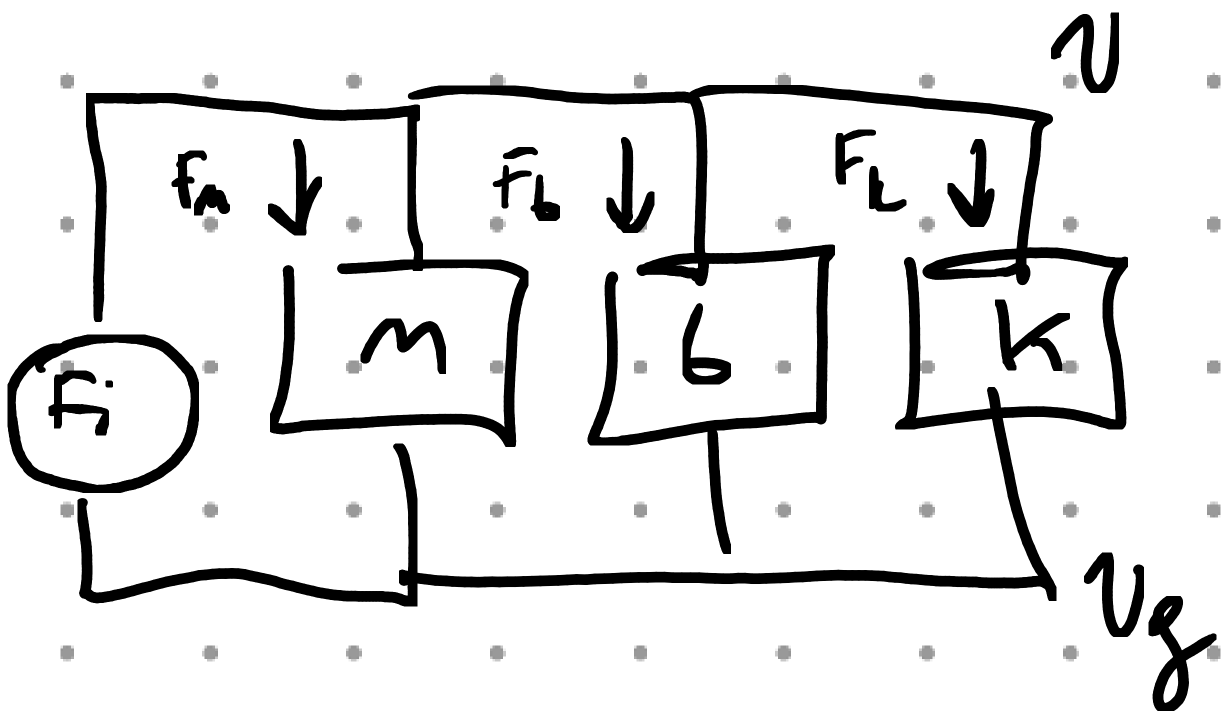
CONSIDER A SIMPLE SYSTEM YOU MIGHT SCOPE FOR A MODEL IN ES103.





$E_m = \frac{1}{2}mv^2$ $E_k = \frac{1}{2k}F_k^2 = \frac{1}{2}kx^2$

IN ES103, WE WOULD ANALYZE USING AN EQUIV CIRCUIT



ELEMENT

$\dot{f}_k = kv$ ①

$f_m = m\dot{v}$ ②

$F_b = bv$ ③

NODE

$F_i = F_m + F_b + F_k$ ④

- WE MIGHT CHOOSE $x = \frac{F_k}{k}$ AS OUR OUTPUT, GET $m\ddot{x} + b\dot{x} + kx = F_i$ FOR OUR MODEL, USING THE "MENU" OF ELEMENT-WISE EQNS AND THE NODE EQUATION TO EXPRESS CONNECTIONS BETWEEN ELEMENTS.
- FOR A SIMPLE SYSTEM LIKE THIS WITH ONLY ONE POSITIONAL "DEGREE OF FREEDOM" (DIRECTION OF INDEPENDENT MOVEMENT) "x," WE COULD ALSO HAVE WRITTEN THE FIRST LAW IN DERIVATIVE FORM FOR THE SYSTEM AS A WHOLE. ENERGY _INSIDE_ IS

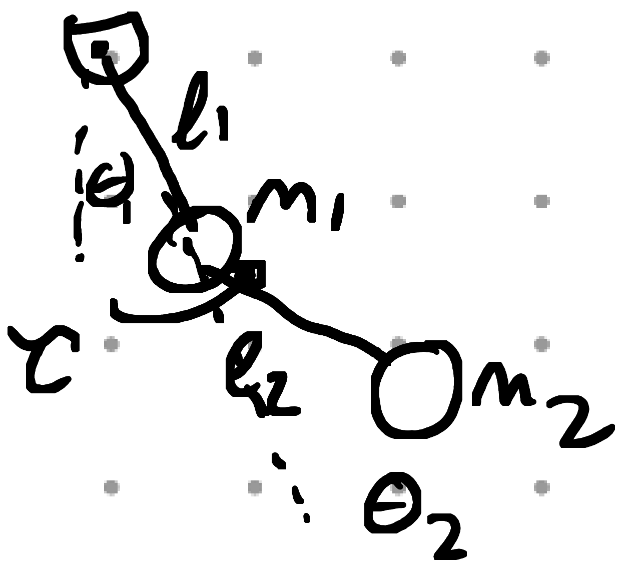
$$E_{TOT} = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

AND WRITING $\dot{E} = \dot{W}_{in} - \dot{Q}_{OUT}$ WITH $\dot{W}_{in} = F_i v = F_i\dot{x}$
AND WITH $\dot{Q}_{OUT} = bv^2 = b\dot{x}^2$, WE COULD WRITE OUR FIRST LAW DERIVATIVE AS

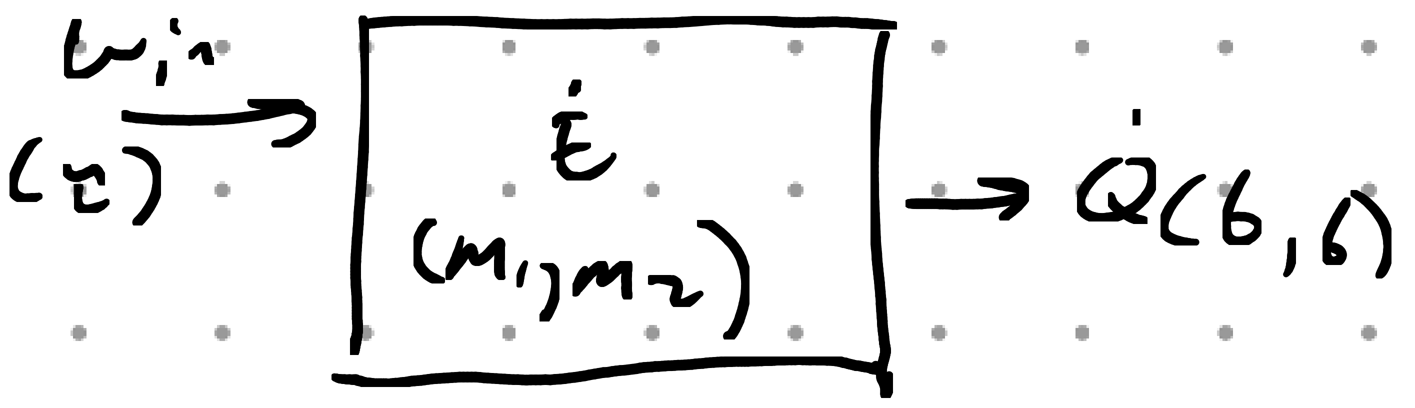
$$m\cancel{\dot{x}}\ddot{x} + kx\cancel{\dot{x}} = F_i\cancel{\dot{x}} - b\dot{x}^{\cancel{2}} \rightarrow m\ddot{x} + b\dot{x} + kx = F_i$$

THIS WORKED BECAUSE OUR EXTRA POWER VARIABLE $\dot{x}$ APPEARS IN ALL TERMS. WILL THIS ALWAYS BE TRUE @ SYSTEM LEVEL?

CONSIDER A SLIGHTLY MORE COMPLEX SYSTEM, WITH 2 "DEGREES OF FREEDOM"



THIS IS A DOUBLE PENDULUM. ASSUME THERE IS DAMPING b AT BOTH HINGES, AND TORQUE τ APPLIED AT HINGE 2 FROM A MOTOR. SLOPING GIVES



M_1, M_2 BOTH STORE KE AND PE.

THIS INDICATES THAT THE SYSTEM IS 4TH ORDER, SINCE EACH MASS REPRESENTS 2 INDEPENDENT ENERGY STORAGE PROCESSES (PE, KE)

$$E_{TOT} = \tfrac{1}{2} M_1 V_1^2 + \tfrac{1}{2} M_2 V_2^2 + M_1 g h_1 + M_2 g h_2$$

h_1 IS DIST ABOVE "RESTING" POSITION FOR M_1
h_2 IS DIST ABOVE "RESTING" POSITION FOR M_2

V_1 IS VELOCITY OF M_1
V_2 IS VELOCITY OF M_2

WRITING THESE IN TERMS OF OUR "DOF" θ_1, θ_2,

$h_1 = (l_1 - l_1 \cos\theta_1)$
$h_2 = l_1 + l_2 - l_1 \cos\theta_1 - l_2 \cos(\theta_1 + \theta_2)$
$V_1 = l_1 \dot{\theta}_1$
$V_2 = l_2 \dot{\theta}_2 + l_1 \dot{\theta}_1$ THEN, WE CAN WRITE E_{TOT} AS

$$E_{TOT} = \tfrac{1}{2}\left[m_1 l_1^2 \dot{\theta}_1^2 + m_2 \left(l_1 \dot{\theta}_1^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 + l_2 \dot{\theta}_2^2\right)\right] \quad (KE)$$

$$+ g\left[m_1 (l_1 - l_1 \cos\theta_1) + m_2 \left(l_1 + l_2 l_1 \cos\theta_1 - l_2 \cos(\theta_1 + \theta_2)\right)\right] (PE)$$

IF WE PROCEED AS BEFORE, AND WRITE THE FIRST LAW'S DERIVATIVE,

$\dot{E} = \dot{W}_{in} - \dot{Q}_{out}$, WE GET:

$$m_1 l_1^2 \dot{\theta}_1 \ddot{\theta}_1 + m_2 \left(l_1 \dot{\theta}_1 \ddot{\theta}_1 + l_1 l_2 \ddot{\theta}_1 \dot{\theta}_2 + l_1 l_2 \dot{\theta}_1 \ddot{\theta}_2 + l_2 \dot{\theta}_2 \ddot{\theta}_2 \right)$$
$$+ g\left(m_1 l_1 \sin\theta_1 \dot{\theta}_1 + m_2 l_2 \sin(\theta_1 + \theta_2) \dot{\theta}_1 + m_2 l_2 \sin(\theta_1 + \theta_2) \dot{\theta}_2\right)$$
$$= \tau_i \dot{\theta}_1 - b\dot{\theta}_1^2 - b\dot{\theta}_2^2$$

THIS DOES NOT HELP US SO MUCH HERE, BECAUSE WE CANNOT CANCEL $\dot{\theta}_1$ <u>OR</u> $\dot{\theta}_2$ FROM EACH TERM! HOW DO WE DEAL WITH THIS? CAN WE SEPARATE $\dot{\bar{E}}$ INTO $\dot{\theta}_1$ AND $\dot{\theta}_2$ PARTS?

IN GENERAL, WE COULD WRITE THE FLT FOR A PURELY MECH SYS AS

$$\underset{\substack{\uparrow\\ \text{E STORED}\\ \text{(KE, PE)}}}{\Delta E} = \underset{\substack{\uparrow\\ \text{WORK } \underline{\text{IN}}}}{\Delta W} - \underset{\substack{\uparrow\\ \text{HEAT } \underline{\underline{\text{OUT}}}}}{\Delta Q}$$

(EG FROM FRICTION)

$$\Delta E = f(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \quad \text{(KINETIC, POTENTIAL)}$$

$\Delta W - \Delta Q = \sum_{i=1}^{n} P_i q_i$ WHERE P_i IS THE NET FORCE/TQ APPLIED TO SYSTEM IN TRANSLATIONAL OR ROTATIONAL DIRECTION "i."

P_i THEN IS THE RESULTANT OF ANY "NON-CONSERVATIVE" ACTIONS ON SYS.

THE DIRECTION "i" IS A "DEGREE OF FREEDOM" (DOF) IN WHICH THE SYSTEM CAN MOVE → TORQUES/FORCES IN THIS DIRECTION WILL DO WORK/CHANGE STORED ENERGY.

WE KNOW THAT WE CAN TAKE TIME DERIV,

$\dot{E} = \dot{W} - \dot{Q}$ AND WE CAN TAKE A PARTIAL DERIVATIVE:

$$\left.\frac{\partial \dot{E}}{\partial \dot{q}_i} = \frac{\partial \dot{W}}{\partial \dot{q}_i} - \frac{\partial \dot{Q}}{\partial \dot{q}_i}\right\} \text{ BY DEFN, THIS IS } \frac{\partial}{\partial q_i} P_i \dot{q}_i = P_i$$

SO WE CAN SAY THAT FOR ANY DOF i,

$$\frac{\partial \dot{E}}{\partial \dot{q}_i} = P_i$$

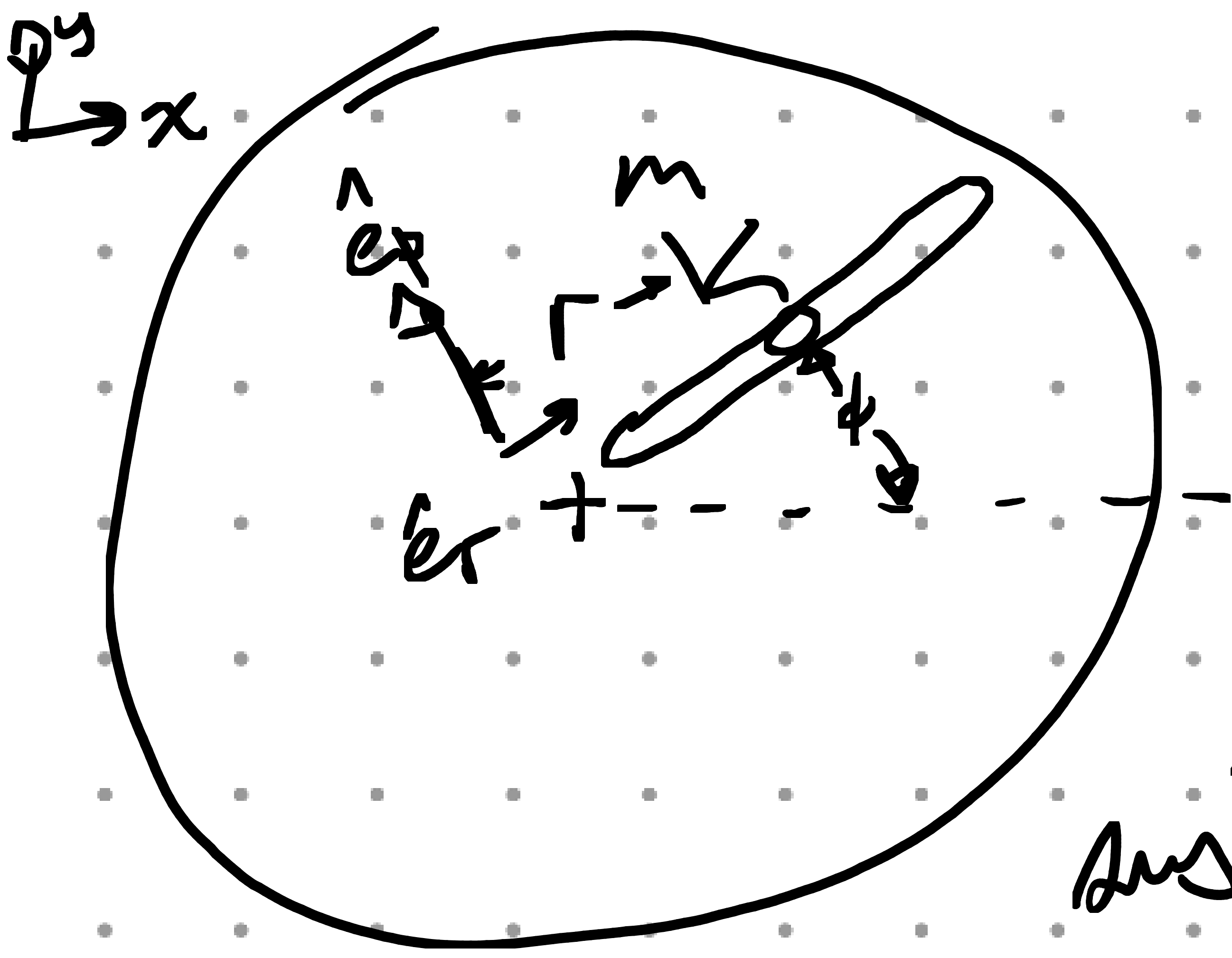
W/ P_i THE NET "GENERALIZED" APPLIED FORCE (COULD BE A TR)

IN THEORY, THIS MEANS WE CAN RECOVER NEWTON'S LAWS AT THE SYSTEM LEVEL FOR "ANY" SYSTEM BY WRITING FLT AT THE SYSTEM LEVEL AND USING

$$\frac{\partial \dot{E}}{\partial \dot{q}_i} = P_i \leftarrow P_i \text{ "NON-CONSERVATIVE" FORCES/TRS.}$$

BUT WILL THIS <u>ALWAYS</u> WORK, FOR <u>ANY</u> DOF CHOICES?

CONSIDER A MARBLE IN A SLOT, FROM AN OVERHEAD VIEW:



LET'S SAY WE WANTED TO CHOOSE "DOF" ϕ, r AND FIND THE EQNS OF MOTION USING FLT.

ASSUME ONLY MARBLE'S MASS IS SIGNIFICANT AND FRICTION IN THE SLOT IS NEGLIGIBLE.

FOR NOW, LET'S SAY THERE ARE NO EXTERNAL APPLIED FORCES OR TORQUES, SO $\Sigma\vec{F}\cdot\hat{e}_r=0$ AND $\Sigma T\cdot\hat{e}_\phi=0$ IN OTHER WORDS, NO APPLIED FORCES WOULD DO ANY WORK ON THE MARBLE, AND IT ONLY STORES $KE=\frac{1}{2}m|v|^2$. GIVEN OUR COORDINATE SYSTEM, WE CAN WRITE:

$\vec{v}=\dot{r}\hat{e}_r+r\dot{\phi}\hat{e}_\theta$; TO SEE WHAT NEWTON 2 SAYS, NOTE THAT TO APPLY $\Sigma\vec{F}=m\vec{a}$ OR $\Sigma T=J\dot{\vec{\omega}}$, WE NEED TO KNOW $\vec{a}$.

$$\vec{a}=\frac{d\vec{v}}{dt}=\ddot{r}\hat{e}_r+\dot{r}\dot{\hat{e}}_r+(r\ddot{\phi}+\dot{r}\dot{\phi})\hat{e}_\phi+r\dot{\phi}\dot{\hat{e}}_\phi$$

NOTE
$$\dot{\hat{e}}_r=\vec{\omega}\times\hat{e}_r=\dot{\phi}\hat{k}\times\hat{e}_r=\dot{\phi}\hat{e}_\phi$$
$$\dot{\hat{e}}_\phi=\vec{\omega}\times\hat{e}_\phi=\dot{\phi}\hat{k}\times\hat{e}_\phi=-\dot{\phi}\hat{e}_r$$

THEN
$$\vec{a}=(\ddot{r}-r\dot{\phi}^2)\hat{e}_r+(2\dot{r}\dot{\phi}+r\ddot{\phi})\hat{e}_\phi$$

THEN N2L SAYS
$$\Sigma F_r=m(\ddot{r}-r\dot{\phi}^2)$$
$$\Sigma T_\phi=\hat{r}\times m\vec{a}=mr(2\dot{r}\dot{\phi}+r\ddot{\phi})$$

IS THIS WHAT WE'D RECOVER FROM $\frac{\partial\dot{E}}{\partial\dot{r}}=\Sigma F_r$ AND $\frac{\partial\dot{E}}{\partial\dot{\phi}}=\Sigma T_\phi$? LET'S FIND OUT!

$E_{STORED} = \frac{1}{2} m|v|^2$ AND WITH $\vec{v} = \dot{r}\hat{e}_r + r\dot{\phi}\hat{e}_\phi$, $|v|^2 = (\dot{r}^2 + r^2\dot{\phi}^2)$

THEN $E_{STORED} = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\phi}^2)$. BY OUR HYPOTHESIS, WE COULD WRITE

$$\sum F_r = \frac{\partial}{\partial \dot{r}}(\dot{E}_{STORED}) = 0 \; ; \quad \sum T_\phi = \frac{\partial}{\partial \dot{\phi}}(\dot{E}_{STORED}) = 0$$

AND OUR EQNS SHOULD MATCH WHAT WE GOT FROM NEWTON.

$$\dot{E} = m\dot{r}\ddot{r} + mr\dot{r}\dot{\phi}^2 + mr^2\dot{\phi}\ddot{\phi}$$

FUT: $\frac{\partial \dot{E}}{\partial \dot{r}} = m\ddot{r} \boxed{+} mr\dot{\phi}^2 = 0$; $\frac{\partial \dot{E}}{\partial \dot{\phi}} = 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = 0$

NEWTON: $m\ddot{r} \boxed{-} mr\dot{\phi}^2 = 0$; $2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} = 0$

→ THERE IS A NEGATIVE SIGN OUT OF PLACE!

WHY IS THIS? IT'S BECAUSE $\dot{\hat{e}}_\phi = \vec{\omega} \times \hat{e}_\phi = \dot{\phi}k \times \hat{e}_\phi = -\hat{e}_r$! THE CENTRIPETAL ACCELERATION IN OUR COORDINATE SYSTEM OPPOSES THE DIRECTION OF $+\dot{r}$... BUT THIS GETS LOST WHEN WE BEGIN WITH E_{STORED} BECAUSE ENERGY IS A DIRECTIONLESS, SCALAR QUANTITY.

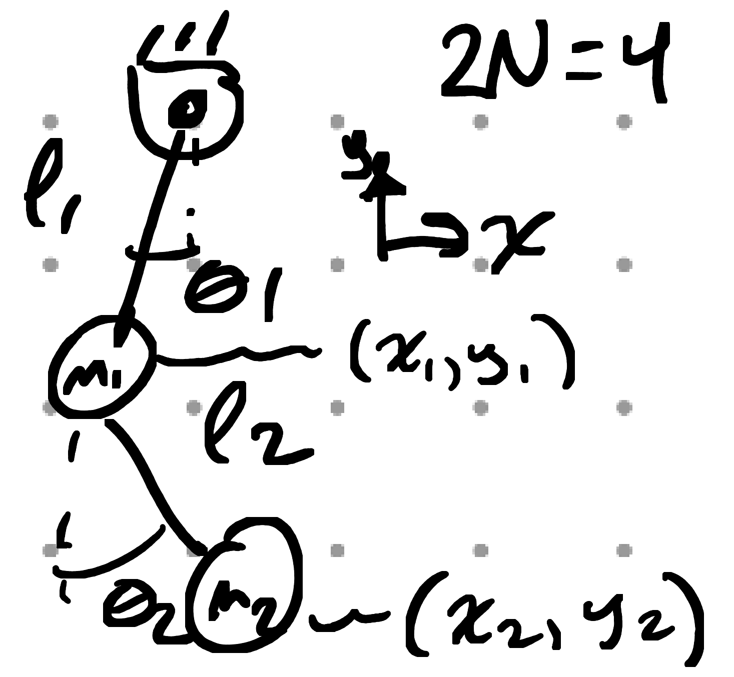
THIS ISNT A PROBLEM IF WE CONSIDER COORDINATE DEFINITIONS THAT ARE "NEWTONIAN" (NON-MOVING) OR WHEN $KE = f(\dot{q}_i)$ BUT NOT $f(\dot{q}_i, q_i)$.

"LAGRANGE'S METHOD" IS SIMILAR TO THE ONE WE JUST DEVELOPED, BUT IT IS ABLE TO HANDLE ANY COORDINATE SYSTEM / DOF DEFINITIONS. LET'S EXPLORE WHERE IT COMES FROM.

DERIVATION BASED ON "CLASSICAL MECHANICS" LIBRETEXT CH 13. (TATUM)

https://phys.libretexts.org/Bookshelves/Classical_Mechanics/Classical_Mechanics_(Tatum)/13%3A_Lagrangian_Mecha

WE CAN SEE WHERE LAGRANGE'S METHOD COMES FROM & HOW IT WORKS BY CONSIDERING ANY MECHANICAL SYSTEM AS A COMPOSITION OF N PARTICLES WITH MASS.
THESE ARE SUBJECT TO k "CONSTRAINTS" THAT LIMIT THEIR MOTION, LEAVING (IN 2D) 2N-k "DEGREES OF FREEDOM."
WE CAN THEN DESCRIBE THE MOTION OF OUR SYSTEM USING 2N-k EQUATIONS OF MOTION IN "GENERALIZED COORDINATES" $\vec{q}$.



2N=4 BUT WE HAVE CONSTRAINTS

$$x_1^2 + y_1^2 = l_1^2$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$$

SO WE ONLY NEED 2 EQNS OF MOTION, PERHAPS WITH
$\vec{q} = [\theta_1, \theta_2]^T$.

MOVING A PARTICLE ALONG θ_1, θ_2 TAKES WORK.

IF WE CALL $\vec{r}_i$ THE POSITION OF THE i^{TH} PARTICLE, AND

$\vec{r}_i$ IS EXPRESSED IN A NEWTONIAN (NON-MOVING) FRAME BUT q_i MIGHT NOT BE, WE CAN STILL ALWAYS WRITE

$$\delta W = \Sigma \underbrace{\vec{F}_i \cdot \delta \vec{r}_i}_{\text{DOT} \to \text{"IN DIRECTION OF"}}$$ WITH $\delta \vec{r}_i$ IN NEWTONIAN COORDS.

WE COULD ALWAYS TRANSFORM $\delta \vec{r}_i$ INTO ANY (J'TH) "GENERALIZED" COORD BY WRITING

$$\delta \vec{r}_i = \frac{\partial \vec{r}_i}{\partial \vec{q}_j} \delta \vec{q}_j$$ SO WE COULD WRITE

$$\delta \vec{W} = \sum_i \vec{F}_i \cdot \sum_j \frac{\partial \vec{r}_i}{\partial \vec{q}_j} \delta \vec{q}_j = \sum_i \sum_j \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial \vec{q}_j} \delta \vec{q}_j$$

BUT IF WE KNOW THAT A SET OF TORQUES/FORCES

$\vec{P}_j$ ACT IN THE DIRECTIONS $\vec{q}_j$, THEN

$\delta\vec{W} = \sum_j \vec{P}_j \cdot \delta\vec{q}_j = \sum_j P_j \delta q_j$ TOO! (P_j IN q_j DIRECTION!)

THIS MEANS THAT USING OUR EQN FOR WORK, WE CAN SAY THAT

$$\delta\vec{W} = \sum_j \boxed{\sum_i F_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}}\, \delta\vec{q}_j = \sum_j \boxed{P_j}\, \delta q_j \rightarrow P_j = \sum_i F_i \frac{\partial \vec{r}_i}{\partial \vec{q}_j}$$

HERE IS WHERE LAGRANGE RELIES ON NEWTONIAN MECHANICS.

NEWTON SAID $\frac{d\vec{p}}{\partial t} = \frac{d}{dt}(m\vec{v}) = \sum \vec{F}$ ON A PARTICLE.

SO WE CAN DO WHAT D'ALEMBERT DID AND REPLACE $\sum\vec{F}$ WITH AN EQUIVALENT "INERTIAL FORCE." THIS MEANS WE CAN WRITE

$$P_j = \sum_i F_i \frac{\partial \vec{r}_i}{\partial \vec{q}_j} = \sum_i m\, \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial \vec{q}_j}$$

HERE IS WHERE THINGS GET STRANGE. WE ARE LOOKING FOR A WAY TO BRING ENERGY INTO THIS... ASSUMING OUR PARTICLES ONLY STORE $KE = \frac{1}{2} m |\vec{v}|^2 = \frac{1}{2} m \dot{\vec{r}} \cdot \dot{\vec{r}}$

WE NOTE THAT BY THE PRODUCT RULE,

$$\frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \underset{\overset{\uparrow}{= \frac{\partial \dot{\vec{r}}_i}{\partial \dot{\vec{q}}_j}}}{\frac{\partial \vec{r}_i}{\partial \vec{q}_j}}\right) = \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial \vec{q}_j} + \underbrace{\dot{\vec{r}}_i \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial \vec{q}_j}\right)}_{= \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j}}$$

SO WE CAN SOLVE FOR $\ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j}$ AND SUB INTO OUR EQN FOR P_j:

$$P_j = \sum_i m_i \left[\frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}\right]$$

LOOKING AT OUR EQN, $P_j = \sum_i m_i \left[\frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} \right) - \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j} \right]$

AND LOOKING AT $KE = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$

WE CAN SEE THAT $\frac{\partial (KE)}{\partial q_j} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$

$$\frac{\partial (KE)}{\partial \dot{q}_j} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

THESE BOTH APPEAR IN OUR P_j EQN SO UPON REPLACEMENT,

$$P_j = \frac{d}{dt}\left(\frac{\partial (KE)}{\partial \dot{q}_j} \right) - \frac{\partial (KE)}{\partial q_j}$$

THIS IS LAGRANGE'S EQN, AND IT CAN BE APPLIED FOR ANY GENERALIZED COORDINATES. LAGRANGE CALLS KE "T."

$$P_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j}$$

NOW, THE P_j CAN BE FROM ANYWHERE, BUT OFTEN SOME ARE FORCES ASSOCIATED W/ THE TRADING OF PE AND KE IN THE SYSTEM. EXAMPLE: IF

$PE = \frac{1}{2} k q^2 + mgq$ THEN THE FORCES ASSOCIATED WITH THE SPRING, GRAVITY CAN BE WRITTEN

$-kq - mg$, OR $\frac{\partial}{\partial q}(PE)$. THEN IF WE CALL $PE = V$ LIKE LAGRANGE, WE CAN WRITE

$P_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j}$ AND NOW P_j ONLY INCLUDES FORCES/TORQUES NOT PART OF SYSTEM'S INTERNAL E. WOULD BE "NON CONSERVATIVE" OR EXTERNALLY APPLIED.

SO LAGRANGE'S METHOD, WHILE RELATED TO THE FIRST LAW, REALLY IS A WAY TO LINK STORED ENERGY TO NEWTON'S EQNS THAT ALLOWS US TO USE ANY "GENERALIZED COORDS" WE WANT. LET'S NOW RETURN TO OUR MARBLE-IN-SLOT.

BY LAGRANGE'S METHOD,

$$\sum F_r = 0 = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot r}\right) - \frac{\partial T}{\partial r} + \cancelto{0}{\frac{\partial V}{\partial r}}$$

$$\sum T_\phi = 0 = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) - \frac{\partial T}{\partial \phi} + \cancelto{0}{\frac{\partial V}{\partial \phi}}$$

WE HAD $T = \frac{1}{2}m(\dot r^2 + r^2\dot\phi^2)$, SO WE COMPUTE:

$$\frac{\partial T}{\partial \dot r} = m\dot r \rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot r}\right) = m\ddot r$$

$$\frac{\partial T}{\partial \dot\phi} = mr^2\dot\phi \rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) = 2mr\dot r + mr^2\ddot\phi$$

$$\frac{\partial T}{\partial r} = mr\dot\phi^2$$

$$\frac{\partial T}{\partial \phi} = 0$$

SO BY LAGRANGE'S EQUATION

$$\left.\begin{aligned} \sum F_r &= 0 = m\ddot r - mr\dot\phi^2 \\ \sum T_\phi &= 0 = 2mr\dot r + mr^2\ddot\phi \end{aligned}\right\} \text{MATCHES NEWTON'S!}$$

THIS SOLVES THE PROBLEM WE HAD WITH USING THE FIRST LAW DIRECTLY. BUT OFTEN, $\frac{\partial T}{\partial q} = 0$ AND EITHER METHOD WOULD WORK; YOU JUST HAVE TO BE CAREFUL! FOR AN EXAMPLE OF WHERE BOTH WORK, SEE OUR DOUBLE PENDULUM SYSTEM.

THE $\frac{d}{dt}$(FLT) FOR THE DOUBLE PENDULUM WAS:

$$m_1 l_1^2 \dot{\theta}_1 \ddot{\theta}_1 + m_2 (l_1^2 \dot{\theta}_1 \ddot{\theta}_1 + l_1 l_2 \ddot{\theta}_1 \dot{\theta}_2 + l_1 l_2 \dot{\theta}_1 \ddot{\theta}_2 + l_2^2 \dot{\theta}_2 \ddot{\theta}_2)$$
$$+ g (m_1 l_1 \sin\theta_1 \dot{\theta}_1 + m_2 l_2 \sin(\theta_1+\theta_2) \dot{\theta}_1 + m_2 l_2 \sin(\theta_1+\theta_2) \dot{\theta}_2)$$
$$= \tau_i \dot{\theta}_1 - b\dot{\theta}_1^2 - b\dot{\theta}_2^2$$

USING $P_j = \frac{\partial \dot{E}}{\partial \dot{q}_j}$ WE WOULD GET

$$\theta_1 : \boxed{\begin{aligned} & m_1 l_1^2 \ddot{\theta}_1 + m_2 (l_1^2 \ddot{\theta}_1 + l_1 l_2 \ddot{\theta}_2) + g m_1 l_1 \sin\theta_1 + g m_2 l_2 \sin(\theta_1+\theta_2) \\ & = \tau_i - b\dot{\theta}_1 \end{aligned}}$$

$$\theta_2 : \boxed{m_2 (l_1 l_2 \ddot{\theta}_1 + l_2 \ddot{\theta}_2) + g m_2 l_2 \sin(\theta_1+\theta_2) = -b\dot{\theta}_2}$$

NOW WITH LAGRANGE, AND USING

$$T = \frac{1}{2}\left[m_1 l_1^2 \dot{\theta}_1^2 + m_2 (l_1^2 \dot{\theta}_1^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \dot{\theta}_2^2) \right] \quad (KE)$$

$$V = g\left[m_1 (l_1 - l_1 \cos\theta_1) + m_2 (l_1 + l_2 - l_1 \cos\theta_1 - l_2 \cos(\theta_1+\theta_2)) \right] \quad (PE)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = m l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + 2 l_1 l_2 \ddot{\theta}_2 M_2 \qquad \frac{\partial T}{\partial \theta_1} = 0$$

$$\frac{\partial V}{\partial \theta_1} = g m_1 l_1 \sin\theta_1 + m_2 l_1 \sin\theta_1 + m_2 l_2 \sin(\theta_1+\theta_2)$$

$$\rightarrow \theta_1 : \boxed{\begin{aligned} & m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 + g (m_1 l_1 \sin\theta_1 + m_2 l_1 \sin\theta_1 + m_2 l_2 \sin(\theta_1+\theta_2)) \\ & = \underbrace{\tau_i - b\dot{\theta}_1}_{P_{\theta_1}} \end{aligned}}$$

P_{θ_1}, EXTERNAL/NONCONSERVATIVE TORQUES IN θ_1 DIRECTION.

SIMILARLY,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + 2 m_2 l_1 l_2 \ddot{\theta}_1$$

$$\frac{\partial T}{\partial \theta_2} = 0 \qquad \frac{\partial V}{\partial \theta_2} = g m_2 l_2 \sin(\theta_1+\theta_2)$$

$$\text{SO } \theta_2 : \boxed{m_2 (l_2^2 \ddot{\theta}_2 + 2 l_2 l_1 \ddot{\theta}_1) + g m_2 l_2 \sin(\theta_1+\theta_2) = -b\dot{\theta}_2}$$

AS YOU CAN SEE, THE RESULTING EQUATIONS OF MOTION ARE THE SAME FOR THIS SYSTEM WHETHER YOU USE LAGRANGE OR THE "EXTENDED $\dot{E}$ LOSS" METHOD, BECAUSE $T \neq f(\vec{q})$!